INITIAL TRANSCRITICAL BEHAVIOR IN A COMPRESSED ROD UNDER CREEP CONDITIONS

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The buckling of a compressed rod in the presence of nonstationary creep is considered. In certain nominal instability criteria [1], the buckling moment is related to the change in rod deflection after the perturbation is applied. The [2] criterion defines the perturbation moment for which the deflection rate is zero as a special point on the time scale separating the conditionally stable deformation from the unstable region. In accordance with it, stability means reduction in deflection at the start of perturbation. Similarly, the deflection acceleration as zero [3] defines a further singular point in the deformation. The distinction of such points has been extended to higher derivatives [4]. The points are called pseudobifurcation ones. In their determination, one equates to zero the derivatives with respect to time of all orders for the deflection, apart from the one corresponding to the pseudobifurcation order.

Here is distinguished singular points in the strain on the basis of the initial transcritical behavior of the higher derivatives without those constraints on the deflection derivatives.

The creep in a hinge-supported rod with length ℓ, area of cross section F, which is compressed by a longitudinal force T is considered. The defining formula is specified, whose general form is derived from the condition for similarity in the creep curves [5]:

$$\dot{p}h(p)=f(\sigma) \tag{1}$$

($p = \varepsilon - \sigma/E$ is the creep strain).

From (1) after linearization we get as follows for small increments in the state of stress—strain corresponding to small deviations from the rectilinear state:

$$\Delta\dot{p}h(p)+\Delta p\dot{p}h'(p)=f'(\sigma)\Delta\sigma.$$

A prime denotes a derivative of a function with respect to its argument. Instead of time t we introduce p:

$$\frac{d(\Delta p)}{dp}+\Delta p\frac{h'(p)}{h(p)}=\frac{f'}{f}\Delta\sigma. \tag{2}$$

We integrate (2) with the initial conditions $\Delta p = \Delta p_0$, $p = p_0$ to get

$$\Delta p=\frac{h(p_0)}{h(p)}\left(\Delta p_0+\frac{f'}{fh(p_0)}\int_{p_0}^{p}\Delta\sigma h(p)\,dp\right). \tag{3}$$

In accordance with the planar-section hypothesis,

$$\Delta\varepsilon=z(w_{xx}-w_{0,xx}), \tag{4}$$

in which w_{xx} is the second derivative of the deflection with respect to the longitudinal coordinate x, while w_0 is the initial deflection derived by perturbation at time $p = p_0$, and z is the transverse coordinate. We put

$$\Delta p=zv_{xx} \tag{5}$$

(v is a function of x). We specify the forms of the functions

$$w=U(p)\sin\lambda x,\ w_0=C_0\sin\lambda x,$$
$$v=V(p)\sin\lambda x,\ \lambda=\pi/l. \tag{6}$$

We use (4)-(6) with $\Delta\sigma = E(\Delta\varepsilon - \Delta p)$ to get the stress increments as

$$\Delta\sigma=Ez\lambda^2(V(p)+C_0-U(p))\sin\lambda x. \tag{7}$$

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The equilibrium equation is

$$\int_F \Delta\sigma z dF = -Tw. \tag{8}$$

The latter two equations give

$$(1-\omega)U(p) = V(p) + C_0, \tag{9}$$

in which $\omega = T/T_0$; $T_0 = EI\lambda^2$ is the critical load for an elastic rod. We rewrite (7) on the basis of (9):

$$\Delta\sigma = -Ez\lambda^2 U(p)\omega\sin\lambda x.$$

We substitute this into (3) to get

$$U(p)(1-\omega) - C_0 - \frac{h(p_0)}{h(p)}\left(V_0 + \frac{f'\omega E}{fh(p_0)}\int_{p_0}^{p} U(p)h(p)\,dp\right) = 0 \tag{10}$$

$[V_0 = V(p_0)]$. The solution to (10) is

$$U(p) = \frac{C_0}{1-\omega}\left\{\frac{h(p_0)}{h(p)}\exp(k(p-p_0)) + \frac{\exp(kp)}{h(p)}\int_{p_0}^{p} h'(p)\exp(-kp)\,dp\right\} + \frac{V_0}{1-\omega}\frac{h(p_0)}{h(p)}\exp(k(p-p_0)) \tag{11}$$

$$(k = (f'/f)E\omega/(1-\omega)).$$

With $p = p_0$, from (9) or (11) we get

$$U(p_0) = (C_0 + V_0)/(1-\omega). \tag{12}$$

We calculate the derivatives of the (11) deflection with respect to time. At the start ($p = p_0$), they are

$$\dot U(p_0) = \dot p_0(C_0 k + V_0(k-q))/(1-\omega),$$
$$\ddot U(p_0) = \dot p_0^2(C_0 k(k-2q) + V_0(k^2 - 3qk + 2q^2 - q_1))/(1-\omega), \tag{13}$$

in which $q = h'(p_0)/h(p_0)$; $q_j = d^j q/dp_0^j$; $j = 1, 2, \ldots$. These expressions may be written for any order N as

$$U^{(N)}(p_0) = \dot p_0^N(C_0 D_N + V_0 B_N)/(1-\omega). \tag{14}$$

Here the polynomials D_N and B_N are constructed as follows:

$$D_1 = k, \quad D_N = kD_{N-1} - \sum_{i=1}^{N-1} D_i C_i^N F_{N-i}, \quad N = 2, 3, \ldots,$$
$$B_0 = 1, \quad B_N = kB_{N-1} - \sum_{i=0}^{N-1} B_i C_i^N F_{N-i}, \quad N = 1, 2, \ldots, \quad C_i^N = \frac{N!}{i!(N-i)!}. \tag{15}$$

The auxiliary functions $F_N(p)$ fit the rule

$$F_1 = q, \quad F_{N+1} = F_N' - (N-1)F_N F_1, \quad N = 1, 2, \ldots$$

We can analyze the behavior of the deflection after perturbation from (12)-(15). If the derivatives of the deflection with respect to time are zero, this corresponds to certain singular points on the time scale. When there is no creep strain ($V_0 = 0$), those points are the roots of the polynomials D_N ($N = 1, 2, \ldots$). The common root of the polynomials is zero: $k = 0$. We denote the other roots by k_{DN} to get

$$k_{D2} = 2q, \quad k_{D3} = (5q \pm \sqrt{12q_1 + q^2})/2. \tag{16}$$

The right-hand sides in (16) are dependent on p_0, while the left-hand ones are dependent on the stress. In particular, if $h(p) = p^\alpha$, then $q = \alpha/p_0$, $q_1 = -\alpha/p_0^2$, and then for a given load there is a singular point on the scale or in time.

Similarly, if $C_0 = 0$ (rod without initial curvature), one gets singular points that are roots of the polynomials B_N:

$$k_{B1} = q, \quad k_{B2} = (3q \pm \sqrt{4q_1 + q^2})/2. \tag{17}$$

The point $k_{B1} = q$ corresponds to the Rabotnov–Shesterikov criterion [2] and $k_{D2} = 2q$ to the Kurshin criterion [1].

Consider another situation. Let the deflection at the start be zero for $C_0 \neq 0$ and $V_0 \neq 0$, which is possible if $C_0 = -V_0$ [1]. Zero deflection derivatives at that instant will correspond to special points in the strain. For the derivatives of order N, we have

$$U^{(N)}(p_0) = \dot{p}C_0H_N/(1-\omega),$$

in which $H_N = D_N - B_N$. We write out the first few polynomials H_N:

$$H_1 = q,\ H_2 = kq + q_1 - 2q^2,\ H_3 = k^2q + k(q_1 - 5q^2) - 7qq_1 + q_2 + 6q^3.$$

We represent the roots as

$$k_{H2} = 2q - q_1/q,\ k_{H3} = (5q - q_1/q \pm \sqrt{(q_1/q)^2 + 18q_1 - 4q_2/q + q^2})/2. \qquad (18)$$

The singular points generated by the H_N can be derived in a different way by replacing the initial condition (5). Let internal stresses occur in the form $\Delta\sigma = zS_{xx}$ (S is a function of x) as a result of a certain perturbation at $p = p_0$. We take S as $S = R(p)\sin\lambda x$ and use the equilibrium equation (8) with (6) to get $R(p) = E\omega U(p)$. On the (4) planar section hypothesis,

$$\Delta p = -\lambda^2 z(U(p)(1-\omega) - C_0)\sin\lambda x,$$
$$\Delta p_0 = -\lambda^2 z(R_0(1/\omega - 1)/E - C_0)\sin\lambda x \quad (R_0 = R(p_0)). \qquad (19)$$

We substitute (19) into (3) and isolate the equation for U(p), which is analogous to (10) but with V_0 replaced by $R_0(1/\omega - 1)/E - C_0$. Then instead of (12) and (14) we have the following expressions for the deflection and its derivatives with respect to time of order N for $p = p_0$:

$$U^N(p_0) = \dot{p}_0^N(C_0H_N/(1-\omega) + R_0B_N/(E\omega)).$$

We put $C_0 = 0$ to find the (17) singular points k_{BN}, and with $R_0 = 0$ we find the singular points k_{HN}. In the latter case, as in the derivation of the k_{HN} points, the initial deflection $U(p_0)$ is zero.

We consider a special case of (1): $f = A\sigma^n$, $h(p) = p^\alpha$. We introduce the dimensionless parameter ξ, which performs the function of time and is monotonically related to it: $\xi = pnEF/(T_0 - T)$. The solutions from (16)-(18) are denoted by the same subscripts as k. We have

$$\xi_{B1} = \alpha,\ \xi_{B2} = (3\alpha \pm \sqrt{\alpha^2 - 4\alpha})/2,\ \xi_{D2} = 2\alpha,\ \xi_{D3} = (5\alpha \pm \sqrt{\alpha^2 - 12\alpha})/2,$$
$$\xi_{H2} = 1 + 2\alpha,\ \xi_{H3} = (5\alpha + 1 \pm \sqrt{\alpha^2 - 18\alpha - 3})/2.$$

The values of ξ for the higher derivatives are derived for $\alpha = 1$. To facilitate the calculations, we use the simple recurrence formulas

$$B_0 = 1,\ B_1 = \xi - 1,\ B_N = (1 - 2N)B_{N-1} + \xi^2B_{N-2}.\ D_1 = \xi,\ D_2 = \xi^2 - 2\xi,$$
$$D_{N+1} = -(1 + 2N)D_N + \xi^2D_{N-1} - (-1)^N\xi(2N-3)!!,\ N = 2, 3, \ldots$$

Table 1 gives the minimal positive values of the roots. For odd N, D_N and H_N do not have roots, and the same applies to B_N for N even. The sequences are monotonically increasing and do not have upper bounds. When one passes through a root, the value of the polynomial alters from negative to positive, so the signs of the corresponding derivatives alter. This does not conflict with the behavior of the singular points. The perturbations applied to the rod before a singular point increase less rapidly than those after it. The ξ_{B1} represents the most hazardous point (zero-order pseudobifurcation [4]). Deflection perturbations defined earlier than ξ_{B1} are reduced at the initial instant and are increased if $\xi > \xi_{B1}$.

There are thus singular points in the deformation on the time scale that have a real physical significance. One can identify the detailed relation between these points and the

TABLE 1

N	ξ_{BN}	ξ_{DN}	ξ_1	N	ξ_{BN}	ξ_{DN}	ξ_4
1	1,00	—	—	5	3,65	—	—
2	—	2,00	3,00	6	—	6,00	7,47
3	2,32	—	—	7	4,97	—	—
4	—	4,00	5,24	8	—	8,04	9,71

buckling moment from experiment, as has been done for the pseudobifurcation points [6]. The singular points for N = 4 may be closest to experiment. If there is no hardening ($\alpha = 0$), degeneracy occurs in the ξ_{BN}, ξ_{DN} singular points (as occurs with many stability criteria for creep). This cannot be said about the ξ_{HN} points. For $\alpha = 0$ we have $\xi_{H_2} = 1.00$, $\xi_{H_4} = 1.78$, $\xi_{H_6} = 2.54$.

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